THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2050 Mathematical Analysis (Spring 2018)
Tutorial on Mar 7

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Part I: Infinite series

Section 3.7 gives a brief introduction to infinite series, which is one of the central topics and will be discussed further in MATH2060. However, this part is not so important in MATH2050 and I will summarize all the important things you need to know for now.

The very first thing you should make clear is that an infinite series is closely related to a sequence. Only based on a given sequence (x_n) can we define a series $S = (s_k)$ where $s_k = \sum_{n=1}^{k} x_n$. But you should distinguish a sequence and a series, which have the same meaning in daily use. A series can be seen as a special kind of sequence, where the terms are obtained by adding terms in a sequence. The second place you should pay special attention to is that where the series starts, i.e., the first term of the series.

We use the same notation $S = \sum x_n$ to denote both the series generated by the sequence (x_n) and the value $\lim S$ (provided the series is convergent).

$\sum x_n$ is said to be convergent if the sequence (s_k) converges. Since a series is defined by adding terms continuously, we have the **n-th term test**:

If the series $\sum x_n$ converges, then $\lim_{n\to\infty} x_n = 0$.

Notice that this is only a **necessary** condition and is usually used to show the divergence of a series. For a series to be convergent, we would not only expect x_n to tend to 0, but also require that the speed of converging to 0 is "sufficiently fast".

Unfortunately, there is NO clear demarcation line between the convergent and divergent series. So people developed many criteria to help judge the convergence. Also, you need to know some special series that can be used for the purpose of comparison.

1. **Two special series**

   - The **geometric series** $\sum_{n=0}^{\infty} r^n$ is convergent if and only if $|r| < 1$.
   - The **p-series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if and only if $p > 1$. And the **harmonic series** $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

2. **Convergence criteria for series of positive terms**

   - **Direct Comparison Test**. Let (x_n) and (y_n) be two real sequences and suppose for some $K \in \mathbb{N}$ and constant $M > 0$ we have

$$x_n \le M y_n, \quad \forall n \ge K.$$

Then $\sum y_n$ converges $\Longrightarrow \sum x_n$ converges, $\sum x_n$ diverges $\Longrightarrow \sum y_n$ diverges.

- **Limit Comparison Test**. Suppose $r = \lim_{n\to\infty} \frac{x_n}{y_n}$ exists.

  If $r = 0$, then $\sum y_n$ converges $\Longrightarrow \sum x_n$ converges, $\sum x_n$ diverges $\Longrightarrow \sum y_n$ diverges.
  If $r > 0$, then $\sum y_n$ converges $\Longleftrightarrow \sum x_n$ converges.
  Remark: We usually choose geometric series, p-series as our comparison series to judge the convergence of other series.

- **Cauchy criterion for series**. The series $\sum x_n$ converges if and only if for every $\varepsilon > 0$, there exists $M(\varepsilon) \in \mathbb{N}$ such that whenever $m > n \geq M(\varepsilon)$, it holds that

$$|s_m - s_n| = |x_{n+1} + x_{n+2} + \cdots + x_m| < \varepsilon.$$

- **Ratio test or D'Alembert's criterion**. Suppose $L = \lim_{n\to\infty} \frac{x_{n+1}}{x_n}$ exists, the ratio test states that
  - if $0 \leq L < 1$, then the series $\sum x_n$ converges,
  - if $L > 1$, then the series $\sum x_n$ diverges,
  - if $L = 1$, then ratio test is inconclusive.

- **Root test or Cauchy's criterion**. Suppose $C = \lim_{n\to\infty} \sqrt[n]{x_n}$ exists, the root test states that
  - if $0 \leq C < 1$, then the series $\sum x_n$ converges,
  - if $C > 1$, then the series $\sum x_n$ diverges,
  - if $C = 1$, then root test is inconclusive.

  Remark: Ratio test and root test are in fact comparison tests where the comparison series is the geometric series.

- **Cauchy's condensation test**.
  Refer to **Ex 3.7.15** in the textbook.

- **Integral test**.
  Will be learned in MATH2060.

3. **(Generalization of 3.7.6(f)) Convergence criterion for alternating series**

   A series of the form

$$\sum_{n=0}^{\infty} (-1)^n x_n = x_0 - x_1 + x_2 - x_3 + \cdots$$

   is called an **alternating series** where either all x_n are positive or all x_n are negative.

   The **Leibniz's test** says: if $|x_n|$ decreases monotonically and $\lim_{n\to\infty} x_n = 0$, then the alternating series converges. (In essence Leibniz's test is an application of MCT)

4. **Convergence criterion for general series**

   - **Absolute convergence**. If $\sum |x_n|$ is convergent, then so is $\sum x_n$.
   - **Abel's test** and **Dirichlet's test**. Will be learned in MATH2060.

Part II: other problems.

1. **(Ex 3.7.9)**

   (a) Show that the series $\sum_{n=1}^{\infty} \cos n$ is divergent.

   (b) Show that the series $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is convergent.

   Solution:

   (a) Since $(\cos n)$ does no converge to 0 (refer to **Example 3.4.6 (c)**), the series $\sum_{n=1}^{\infty} \cos n$ is divergent from the n-th term test.

   (b) First notice that $\left|\frac{\cos n}{n^2}\right| \leq \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is p-series $(p = 2 > 1)$ and hence convergent. By Comparison Test we know $\sum_{n=1}^{\infty} \left|\frac{\cos n}{n^2}\right|$ is convergent. Then Absolute Convergence implies that $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is also convergent.

2. **(Ex 3.7.11-12)** Suppose $\sum a_n$ with $a_n > 0$ is convergent, show that $\sum a_n^2$ is also convergent. Give a counterexample to show that the converse is not true.

   Solution: By the n-th term test we know $\lim_{n\to\infty} a_n = 0$ and thus (a_n) is bounded, i.e., there exists some $M > 0$ such that $0 < a_n < M, \forall n$.

   Therefore, $0 < a_n^2 < Ma_n$ and we obtain the convergence of $\sum a_n^2$ from the Comparison Test.

   A counterexample is $a_n = \frac{1}{n}$.

3. **(Ex 3.7.13)** Suppose $\sum a_n$ with $a_n > 0$ is convergent, show that $\sum \sqrt{a_n a_{n+1}}$ is also convergent. Give a counterexample to show that the converse is not true.

   Solution: From the AM-GM inequality we have $\sqrt{a_n a_{n+1}} \leq \frac{a_n + a_{n+1}}{2}$. Then by the Comparison Test we know that $\sum \sqrt{a_n a_{n+1}}$ is convergent.

   A counterexample is given by $a_{2n-1} = 1, a_{2n} = \frac{1}{n^4}, \quad n = 1, 2, 3, \cdots$.

4. **(Ex 3.7.15 Cauchy Condensation Test)**. Let $\sum_{n=1}^{\infty} a_n$ be such that (a_n) is a decreasing sequence of strictly positive numbers and s_n denotes the nth partial sum, show that

$$\frac{a_1 + 2a_2 + \cdots + 2^n a_{2^n}}{2} \leq s_{2^n} \leq a_1 + 2a_2 + \cdots + 2^{n-1} a_{2^{n-1}} + a_{2^n}.$$

   Use these inequalities to show that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} 2^n a_{2^n}$ converges.

Proof: Since (a_n) is decreasing, we have

$$\begin{aligned}
s_{2^n} &= a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \cdots + (a_{2^{n-1}} + \cdots + a_{2^n-1}) + a_{2^n} \\
&\leq a_1 + 2a_2 + 4a_4 + \cdots + 2^{n-1}a_{2^{n-1}} + a_{2^n} \\
&\leq a_1 + 2a_2 + 4a_4 + \cdots + 2^{n-1}a_{2^{n-1}} + 2^n a_{2^n}, \\
s_{2^n} &= a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + \cdots + (a_{2^{n-1}+1} + \cdots + a_{2^n}) \\
&\geq a_1 + a_2 + 2a_4 + \cdots + 2^{n-1}a_{2^n} \\
&\geq \frac{a_1 + 2a_2 + \cdots + 2^n a_{2^n}}{2}.
\end{aligned}$$

By Comparison Test, the conclusion follows. (Please supplement all the details yourself)

Part III: Additional exercises.

1. **(Ex 3.7.18)** Show that if $c > 1$, then the following series are convergent:

   (a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^c}$,

   (b) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n \ln n (\ln\ln n)^c}$. (Hint: use (a) and Limit Comparison test)

   Solution:

   (a) Using the Cauchy Condensation Test where $a_n = \dfrac{1}{n(\ln n)^c}$, we have

   $$2^n a_{2^n} = 2^n \cdot \frac{1}{2^n (\ln 2^n)^c} = \frac{1}{(n \ln 2)^c} = \frac{1}{(\ln 2)^c} \cdot \frac{1}{n^c}.$$

   Since $\sum \dfrac{1}{n^c}$ is a p-series where $p = c > 1$ and consequently convergent, we know that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^c}$ is also convergent.

   (b) This time

   $$2^n a_{2^n} = 2^n \cdot \frac{1}{2^n \ln 2^n (\ln\ln 2^n)^c} = \frac{1}{n \ln 2 [\ln(n \ln 2)]^c} = \frac{1}{n \ln 2 (\ln n + \ln\ln 2)^c}.$$

   Use the Limit Comparison Test and we have

   $$\lim_{n\to\infty} \frac{\frac{1}{n \ln 2 (\ln n + \ln\ln 2)^c}}{\frac{1}{n(\ln n)^c}} = \lim_{n\to\infty} \frac{1}{\ln 2} \left(\frac{\ln n}{\ln n + \ln\ln 2} \right)^c = \frac{1}{\ln 2}$$

   which is positive. Therefore, $\sum \dfrac{1}{n \ln 2 (\ln n + \ln\ln 2)^c}$ and $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^c}$ are simultaneously convergent or divergent.

   From the result in (a) we know that $\sum \dfrac{1}{n \ln 2 (\ln n + \ln\ln 2)^c}$ is convergent and the Cauchy Condensation Test implies that $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n \ln n (\ln\ln n)^c}$ is also convergent.

2. Suppose both the series $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ are convergent. Show that

   (a) $\sum_{n=1}^{\infty} x_n y_n$ is convergent,

   (b) $\sum_{n=1}^{\infty} \frac{x_n}{n}$ is convergent.

   Proof: (a) By **Ex 3.7.11** we have that both $\sum x_n^2$ and $\sum y_n^2$ are convergent. Then from the Comparison Test, $\sum x_n y_n$ is convergent since $x_n y_n \leq \dfrac{x_n^2 + y_n^2}{2}$.

   (b) $\dfrac{x_n}{n} \leq \dfrac{x_n^2}{2} + \dfrac{1}{2n^2}$ by AM-GM inequality and both of $\sum_{n=1}^{\infty} \dfrac{x_n^2}{2}$, $\sum_{n=1}^{\infty} \dfrac{1}{2n^2}$ are convergent.